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Coherent neutron fields and the Lie algebra $sl(2, R)$

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Abstract. The application of Glauber's definition of quantum coherence allows the introduction of neutron fields of partial coherence under the assumption that the complete occupation number space is a direct product of Fermi subspaces. Excitations are described by 'collective' creation and annihilation operators which span an algebra isomorphic to the $sl(2, R)$ algebra. The associated coherent states are of partial coherence and the finite dimensional representation of the $su(2, R)$ algebra is not regularly coherent. In contrast, the transition to an infinite dimensional representation space results in regular coherent properties of the field. This is demonstrated using the representation space of the $su(1, 1)$ algebra which has a real isomorphism to the $sl(2, R)$ algebra. The coherent states calculated from Glauber's condition for coherence are completely coherent, and are, moreover, identical to those found by Barut and Girardello [9] in starting from a far more abstract argument.

1. Introduction

Coherence describes, in classical field theory, the ability of spacetime interference at different points. Wolf's correlation functions [1] are used to describe this property of electromagnetic fields. In case of laser light and other 'non-natural' sources an expansion to higher order correlation functions is necessary. Quantized fields on the other hand, with their emphasis on the particle aspect do not, at first sight, fit into such a picture of coherence.

Nevertheless, it is proved to be possible to describe coherent particle fields inside the framework of quantum electrodynamics using correlation functions [2, 3, 9]. In this case coherence is defined by a *random* spacetime coincidence of field particles and this property is described by a factorization rule of the nth order correlation function $G^{(n)}(x_1, \dots, x_n; x_n, \dots, x_1)$:

$$\begin{aligned} G^{(n)}(x_1, \dots, x_n; x_n, \dots, x_1) &= Tr\{\rho E^{(-)}(x_1) \dots E^{(-)}(x_n) E^{(+)}(x_n) \dots E^{(+)}(x_1)\} \\ &= \langle| E^{(-)}(x_1) \dots E^{(-)}(x_n) E^{(+)}(x_n) \dots E^{(+)}(x_1) |\rangle \\ &= \prod_{i=1}^{n} \langle| E^{(-)}(x_i) E^{(+)}(x_i) |\rangle; \; n = 1, 2, \dots, \infty \end{aligned} \tag{1}$$

where $E^{(-)}$ and $E^{(+)}$ are, respectively, photon creation and annihilation operators; ρ is the density matrix

$$\rho = |\rangle\langle|. \tag{2}$$

Each state $|\alpha\rangle$ which satisfies equation (1) represents a coherent state of the photon field. We speak of *complete* coherence if the states $|\alpha\rangle$ satisfy equation (1) even for $n \to \infty$; on the other hand, if equation (1) can only be fulfilled for finite n, we speak of *partial* coherence of nth order.

It is most satisfying that such a definition of coherence is in full agreement with the definition of coherence in classical field theory. In cases of well defined (i.e. a fixed phase correlation exists between different space points of the field) the number of particles is undetermined because of the uncertainty principle between the number of particles and the field strength. Consequently, we have no correlation between particles at different space points except by accidental processes.

Other authors [5–8] discussed expansions of the theory of quantum coherence using the fact that coherent states $|\alpha\rangle$ of boson fields are equivalently described by

(i) the solutions of the eigenvalue equation

$$b|\alpha\rangle = \alpha|\alpha\rangle; \qquad \langle\alpha|b^\dagger = \alpha^*\langle\alpha| \tag{3}$$

with α an arbitrary complex number, and the Heisenberg operators b and $b^\dagger$ which obey the commutation rules

$$[b, b] = [b^\dagger, b^\dagger] = 0; \qquad [b, b^\dagger] = E. \tag{4}$$

(ii) applying the operator

$$D(\alpha) = \exp\{-\tfrac{1}{2}|\alpha|^2\} \exp\{\alpha b^\dagger\} \tag{5}$$

on the vacuum state $|0\rangle$ which is defined by

$$b|0\rangle = 0. \tag{6}$$

Such an operator generates all possible coherent states $|\alpha\rangle$ which are described by

$$|\alpha\rangle = \exp\{-\tfrac{1}{2}|\alpha|^2\} \sum_{m=0}^{\infty} \frac{\alpha^m}{m!} |m\rangle. \tag{7}$$

It is certainly of interest to extend the definition of coherence according to equation (3) or (5) by replacing the Heisenberg algebra $\{b, b^\dagger, E\}$ by a more structured Lie algebra of the $su(p, q)$ type which is so important in elementary particle physics. Barut and Girardello [5], for instance, derived general coherent states based on equation (3) using the elements of the noncompact algebra $su(1, 1)$. Definition (5) was also used to define generalized coherent states using more general Lie algebras [6, 7].

Glauber's factorization rule (1) was mainly developed for bosonic systems; nevertheless, it was possible to expand it to fermionic systems [4, 10–12]. It was one of the key results of such an expansion [4] that it was not possible to factorize the correlation function of nth order, with the result that there are no coherent Fermion fields in the sense of equation (1). Only the introduction of uniformly correlated states [10] which obey a Poisson distribution in the limit of infinite particle numbers allowed the construction of coherent Fermion fields. One application of this rather abstract concept was discussed by Ledinegg and Schachinger [11]. They considered electrically neutral fermions emitted by a *stochastic* source, for instance neutrons emitted by a reactor core which can be collimated to form a neutron beam. In such a system we have no causal relations between the various neutron creation processes and this is expressed mathematically by a state space $\mathcal{R}$ which is the direct product of Fermi subspaces $\mathcal{R}^{(n)}$:

$$\mathcal{R} = \mathcal{R}^{(1)} \otimes \mathcal{R}^{(2)} \otimes \cdots \otimes \mathcal{R}^{(n)}. \tag{8}$$

In such a model, coherent neutron states are defined in analogy to equation (1) as the solutions of the factorization rule for the n-th order correlation function:

$$\begin{aligned} G^{(n)}(x_1, \ldots, x_n; x_n, \ldots, x_1) &= \mathrm{Tr}\{\rho\hat{\psi}^{(-)}(x_1) \ldots \hat{\psi}^{(-)}(x_n)\hat{\psi}^{(+)}(x_n) \ldots \hat{\psi}^{(+)}(x_1)\} \\ &= \langle|\hat{\psi}^{(-)}(x_1) \ldots \hat{\psi}^{(-)}(x_n)\hat{\psi}^{(+)}(x_n) \ldots \hat{\psi}^{(+)}(x_1)|\rangle \\ &= \prod_{i=1}^{n} \langle|\hat{\psi}^{(-)}(x_i)\hat{\psi}^{(+)}(x_i)|\rangle. \end{aligned} \tag{9}$$

The operators $\hat{\psi}^{(-)}(x)$ and $\hat{\psi}^{(+)}(x)$ are field operators which are built from the original Fermi creation and annihilation operators $a^\dagger$ and a by means of 'collective' creation and annihilation operators $\hat{a}^\dagger(\boldsymbol{q})$ and $\hat{a}(\boldsymbol{q})$ [11]. They increase (decrease) the particle number of 'uniformly correlated states' $|m\rangle_q \in \mathcal{R}$ which describe m neutrons of momentum $\boldsymbol{q}$ in their respective subspaces $(m \leq n)$. The action of $\hat{a}^\dagger(\boldsymbol{q})$ and $\hat{a}(\boldsymbol{q})$ on the states $|m\rangle_q$ is described by

$$\begin{aligned} \hat{a}(\boldsymbol{q})|m\rangle_q &= \sqrt{m}\, f_1(n; m)|m-1\rangle_q \\ \hat{a}^\dagger(\boldsymbol{q})|m\rangle_q &= \sqrt{m+1}\, f_2(n; m)|m+1\rangle_q \end{aligned} \tag{10}$$

with

$$f_1(n; m) = \left(\frac{n-m+1}{n}\right)^{1/2} \qquad f_2(n; m) = f_1(n; m+1) \tag{11}$$

(equation (6–8) of Ref. 10). In the limit

$$\lim_{n\to\infty} f_1(n; m) = \lim_{n\to\infty} f_2(n; m) = 1, \tag{12}$$

the operators $\hat{a}^\dagger(\boldsymbol{q})$ and $\hat{a}(\boldsymbol{q})$ become standard Bose creation and annihilation operators while they behave entirely like Fermi operators in the limit $n \to 1$.

It is the inclination of this paper to discuss some group theoretical aspects of the above model for neutron coherence. We can do so, as Ledinegg and Schachinger already discussed in great detail the physical implications of such a system in calculating the coherence time or spacetime correlations [11, 12]. The group theoretical aspects are developed by showing in chapter II that the collective creation and annihilation operators span a $sl(2, R)$ algebra. In chapter III coherent states are derived using the factorization rule (9) and it is shown that these states are of partial coherence only. On the other hand, it is also shown that the eigenvalue equation (3) does not have a solution.

This is the result of a finite representation space which is based on the collective neutron states which were constructed from a physical argument. It is of course also possible to relate the step operators of the $sl(2, R)$ algebra to an infinite representation space. Such a representation allows again, naturally, the introduction of coherent states using either the eigenvalue equation (3) or the factorization rule (9). It is now appropriate to use the step operators of the $su(1, 1)$ algebra which has a real isomorphism with the $sl(2, R)$ algebra. Coherent states of the $su(1, 1)$ algebra have already been calculated by Barut and Girardello [5] using the eigenvalue equation (3). Chapter IV then shows that Glauber's factorization rule results in completely coherent states and that these states are identical to the ones found by Barut and Girardello. This is a fine example to demonstrate how the coherence properties of a system change if the set of eigenfunctions of the Cartan subalgebra belongs to different representations of a given Lie algebra.

2. Collective creation and annihilation operators and their Lie algebra

Each Fermi subspace $\mathscr{R}^{(\nu)} \in \mathscr{R}$ (cf equation (8)) allows only states of occupation number 0 or 1, i.e. only free Fermi particles of identical momentum $\boldsymbol{q}$, fixed bispinor component and fixed spin orientation are possible. If we assume these Fermi particles to have the same *a priori* probability in their respective subspaces, states $|m\rangle_q$ of a certain number of particles $(m \leqslant n)$ which may be detected simultaneously can be constructed as

$$|m\rangle_q = \binom{n}{m}^{1/2} \sum_{\{\nu_1 \ldots \nu_m\}} {}^{(1)}|0\rangle_q \ldots {}^{(\nu_1)}|1\rangle_q \ldots {}^{(\nu_m)}|1\rangle_q \ldots {}^{(n)}|0\rangle_q \tag{13}$$

where $\{\nu_1 \ldots \nu_m\}$ denotes the set of possible combinations of m occupied subspaces in the total number of n subspaces. ${}^{(\nu)}|0\rangle_q$ and ${}^{(\nu)}|1\rangle_q$ are the two eigenstates of the ν-th Fermi subspace. The set of orthonormal states $\{|m\rangle_q,\ m=0, 1, \ldots, n\}$ spans the state space $\mathscr{R}_F$.

Experimentally, the correlation function is measured by particle detectors which are placed at different space points. Each detector absorbs particles of each subspace $\mathscr{R}^{(\nu)}$, a feature we would like to express by 'collective' annihilation and creation operators, [11] $\hat{a}(\boldsymbol{q})$ and $\hat{a}^{\dagger}(\boldsymbol{q})$ respectively. They are defined by

$$\begin{aligned} \hat{a}(\boldsymbol{q}) &= \frac{1}{\sqrt{n}} \sum_{\nu=1}^{n} a_\nu(\boldsymbol{q}) \\ \hat{a}^{\dagger}(\boldsymbol{q}) &= \frac{1}{\sqrt{n}} \sum_{\nu=1}^{n} a_\nu^{\dagger}(\boldsymbol{q}) \end{aligned} \tag{14}$$

where the $a_\nu(\boldsymbol{q})$ and $a_\nu^{\dagger}(\boldsymbol{q})$ are standard Fermi annihilation and creation operators acting in the subspace $\mathscr{R}^{(\nu)}$:

$$a_\nu(\boldsymbol{q})^{(\nu)}|1\rangle_q = {}^{(\nu)}|0\rangle_q \qquad a_\nu{}^{(\nu)}|0\rangle_q \equiv 0 \tag{15}$$

$$a_\nu^{\dagger}(\boldsymbol{q})^{(\nu)}|0\rangle_q = {}^{(\nu)}|1\rangle_q \qquad a_\nu^{\dagger(\nu)}|1\rangle_q \equiv 0. \tag{16}$$

They observe the usual anticommutation rules in their respective subspaces. To express the fact that in general there will be no Pauli exclusion between independent Fermi absorbers, operators belonging to different subspaces are supposed to commute with each other:

$$[a_\nu(\boldsymbol{q}), a_\mu(\boldsymbol{q})] = [a_\nu^{\dagger}(\boldsymbol{q}), a_\mu^{\dagger}(\boldsymbol{q})] = [a_\nu(\boldsymbol{q}), a_\mu^{\dagger}(\boldsymbol{q})] = 0 \qquad \nu \neq \mu \tag{17}$$

with

$$[a, b] = ab - ba.$$

From these definitions it becomes transparent how the physical properties of the detectors used in a coherence experiment will affect its outcome.

We can expect from equations (15)–(17) the collective creation and annihilation operators to span an algebra isomorphic to the $sl(2, R)$ algebra. Therefore, we only have to calculate the explicit form of the commutation rules.

Equations (10) describe the action of the collective operators on the states $|m\rangle_q$ and these equations establish the basis from which the Lie algebra of the collective operators can be determined. Obviously, the Lie product

$$[\hat{a}, \hat{a}^{\dagger}] = \hat{a}\hat{a}^{\dagger} - \hat{a}^{\dagger}\hat{a} \tag{18}$$

is neither proportional to the unity operator nor to a linear combination of $\hat{a}$ and $\hat{a}^\dagger$. Thus, we define the operator

$$\hat{b} = [\hat{a}, \hat{a}^\dagger] \tag{19}$$

where we have dropped the explicit $\boldsymbol{q}$ argument in the understanding that from now on all operators are assumed to be of the same argument. As each state $|m\rangle \in \mathcal{R}_F$ is an eigenstate of $\hat{a}\hat{a}^\dagger$ and of $\hat{a}^\dagger\hat{a}$, $|m\rangle$ is also an eigenstate of $\hat{b}$:

$$\hat{b}|m\rangle = \left(1 - \frac{2m}{n}\right)|m\rangle. \tag{20}$$

To complete the argument, the action of the Lie-products $[\hat{a}, \hat{b}]$ and $[\hat{a}^\dagger, \hat{b}]$ on the state $|m\rangle$ is calculated:

$$[\hat{a}, \hat{b}]|m\rangle = -\frac{2}{n}\sqrt{m\left(1 - \frac{m-1}{n}\right)}\,|m-1\rangle = -\frac{2}{n}\,\hat{a}|m\rangle \tag{21}$$

$$[\hat{a}^\dagger, \hat{b}]|m\rangle = -\frac{2}{n}\sqrt{(m+1)\left(1 - \frac{m}{n}\right)}\,|m+1\rangle = \frac{2}{n}\,\hat{a}^\dagger|m\rangle. \tag{22}$$

This follows immediately if the states $|m-1\rangle$ and $|m+1\rangle$ are rewritten using equations (10).

The relations (19), (21) and (22) define a Lie algebra $\{c\}$ of the form

$$c = \alpha\hat{a} + \beta\hat{a}^\dagger + \gamma\hat{b} \tag{23}$$

with

$$[\hat{a}, \hat{b}] = -\frac{2}{n}\,\hat{a} \qquad [\hat{a}^\dagger, \hat{b}] = \frac{2}{n}\,\hat{a}^\dagger \qquad [\hat{a}, \hat{a}^\dagger] = \hat{b} \tag{24}$$

and the arbitrary real numbers α, β, and γ. A basic transformation

$$\frac{n}{2}\,\hat{b} = -J_0 \qquad \sqrt{n}\,\hat{a} = J_- \qquad \sqrt{n}\,\hat{a}^\dagger = J_+ \tag{25}$$

results in the commutation rules

$$[J_0, J_\pm] = \pm J_\pm \qquad [J_+, J_-] = 2J_0 \tag{26}$$

which generate, as expected, the real Lie algebra *sl*(2, *R*) [13]. Furthermore, we have

$$\begin{aligned} J_0|j; m'\rangle &= m'|j; m'\rangle \\ J_+|j; m'\rangle &= \sqrt{j(j+1) - m'(m'+1)}\,|j; m'+1\rangle \\ J_-|j; m'\rangle &= \sqrt{j(j+1) - m'(m'-1)}\,|j; m'-1\rangle \end{aligned} \tag{27}$$

with $m' = m - j$ and $j = n/2$.

3. Coherent neutron states associated with the *sl*(2, *R*) algebra

It was already pointed out by Glauber that the factorization rule (1) and the eigenvalue equation (3) are equivalent conditions for *completely* coherent states if the creation and annihilation operators span a Heisenberg algebra. To show this, he employed the eigenvalue equation

$$E^{(+)}(x)|\alpha\rangle = \varepsilon(x)|\alpha\rangle \tag{28}$$

which is equivalent to equation (3) and compared its results to the coherent states found from solving (1) for $n \to \infty$.

This fact suggests the definition of a *regular coherent* representation of an algebra. Such a representation has the significant property that identical coherent states are found by either solving the factorization rule or by solving the eigenvalue equation of the step operators.

The obvious next step is then to investigate whether the finite dimensional (non-unitary) representation of the $sl(2, R)$ algebra introduced by equation (27) is regular coherent or not. We, therefore, start with the eigenvalue equation

$$J_-|\alpha\rangle = \alpha|\alpha\rangle \tag{29}$$

and use the ansatz

$$|\alpha\rangle = \sum_{j=1/2}^{\infty} \sum_{m=-j}^{j} C_{m,j}|j; m\rangle. \tag{30}$$

The coherent states are found from

$$\begin{aligned} J_-|j, m\rangle &= \sqrt{j(j+1) - m(m-1)}\,|j; m-1\rangle \\ &= g(j, m)|j; m-1\rangle \end{aligned} \tag{31}$$

and

$$J_-|j; -j\rangle \equiv 0 \qquad J_+|j, j\rangle \equiv 0. \tag{32}$$

This results in the recursion relation

$$C_{j,m} = \alpha g^{-1}(j, m) C_{j,m-1}. \tag{33}$$

Equation (32) requires $C_{j,j} \equiv 0$ with the consequence that all the other coefficients $C_{j,m}$ ($m \neq j$) are zero as well and the system has no states of complete coherence.

Nevertheless coherent neutron states can be found from Glauber's factorization rule (9). To show this, we define in analogy to [10], equation (11), the spinor field operators $\psi^{(+)}$ and $\psi^{(-)}$ as

$$\begin{aligned} \psi^{(+)}(x) &= f_q(x) J_- \\ \psi^{(-)}(x) &= f_q^*(x) J_+ \end{aligned} \tag{34}$$

with

$$f_q(x) = \exp\{i(\boldsymbol{q}\boldsymbol{x} - E x_0)\} u_{r,\alpha}(q). \tag{35}$$

$x = (x_0, \boldsymbol{x})$ is the four-dimensional 'spacetime' vector, $u_{r,\alpha}$ is the bispinor component of polarization r and a fixed bispinorindex α, ($\alpha, r = 1, 2, 3, 4$) and $q = (E_0, \boldsymbol{q})$. Transformation of equation (30) to occupation space results in [4, 10, 11]

$${}_q\langle\, |^{\overbrace{q\ldots q}^{s}}\, |\rangle_q^{\overbrace{q\ldots q}^{s}} = \{{}_q\langle\, |^q\, |\rangle_q^q\}^s \qquad 1 \leqslant s \leqslant 2j \tag{36}$$

with the states

$$\begin{aligned} |\rangle_q &= \sum_{j=1/2}^{\infty} \sum_{m=-j}^{j} C_{j,m}|j; m\rangle \\ \langle|_q &= \sum_{j=1/2}^{\infty} \sum_{m=-j}^{j} C_{j,m}^*\langle j; m| \end{aligned} \tag{37}$$

the 'right-hand operation'

$$| \rangle_q^q = J_- | \rangle_q = \sum_{j=1/2}^{\infty} \sum_{m'=-j+1}^{j} C_{j,m'} f_1(j; m') |j; m'-1\rangle_q \tag{38}$$

and the 'left-hand operation'

$$^q| \rangle_q = J_+ | \rangle_q = \sum_{j=1/2}^{\infty} \sum_{m'=-j}^{j-1} C_{j,m'} f_2(j; m') |j; m'+1\rangle_q. \tag{39}$$

If both operations are to be performed as it is the case in equation (36), the right-hand operation is performed first. We find for instance:

$$_q\langle |^q | \rangle_q^q = \sum_{j=1/2}^{\infty} \sum_{m'=-j+1}^{j} |C_{j,m'}|^2 f_1^2(j; m') \tag{40}$$

or

$$_q\langle |^{\overbrace{q \dots q}^{s}} | \rangle_q^{\overbrace{q \dots q}^{s}} = \sum_{j=1/2}^{\infty} \sum_{m'=-j+s}^{j} |C_{j,m'}|^2 \prod_{\nu=0}^{s-1} f_1^2(j; m'-\nu). \tag{41}$$

These results are used in equation (36) and it is sufficient to study it for a fixed value of j as j_- and J_+ do not change the subspace $\{|j; m'\rangle\}$ according to equation (31). We find

$$\sum_{m'=-j+s}^{j} |C_{j,m'}|^2 \prod_{\nu=0}^{s-1} f_1^2(j, m'-\nu) = \alpha^{2s} \left\{ \sum_{m'=-j+1}^{j} |C_{j,m'}|^2 f_1^2(j, m') \right\}^s \qquad s = 1, 2, \dots, 2j \tag{42}$$

with

$$\alpha^2 = \sum_{m'=-j+1}^{j} |C_{j,m'}|^2 f_1^2(j, m'). \tag{43}$$

The coefficients

$$a_{s,r} = \prod_{\nu=0}^{s-1} f_1^2(j, m'-\nu); \; r = m'+j \tag{44}$$

on the left hand side of equation (42) are elements of an upper triangle matrix $A(j)$ of dimension $2j$. Such a representation simplifies the solution of equation (42) for the coefficients $|C_{j,m}|^2$ because this equation can be transformed into

$$\begin{pmatrix} |C_{j,-j+1}|^2 \\ \vdots \\ |C_{j,j}|^2 \end{pmatrix} = \frac{1}{|A_{ij}|} \begin{pmatrix} A_{11} & 0 & \dots & 0 \\ A_{12} & A_{22} & \dots & 0 \\ \dots & \dots & \dots & \dots \\ A_{1j} & \dots & \dots & A_{jj} \end{pmatrix} \begin{pmatrix} \alpha^2 \\ \alpha^4 \\ \dots \\ \alpha^{4j} \end{pmatrix}. \tag{45}$$

The $A_{j,k}$ are the algebraic complements of the $a_{j,k}$ from equation (44). The remaining coefficient $C_{j,-j}$ is determined from the normalization condition $\langle\alpha|\alpha\rangle=1$ and Glauber's factorization rule indeed results in coherent states

$$|\alpha\rangle_j=\sum_{m=-j+1}^{j} C_{j,m}|j, m\rangle+C_{j,-j}|j,-j\rangle \tag{46}$$

with

$$C_{j,m}=\frac{1}{|A_{ij}|^{1/2}}\left\{\sum_{i=1}^{s} A_{is}\alpha^{2i}\right\}^{1/2} \tag{47}$$

and

$$|C_{j,-j}|^2=1-\sum_{m=-j+1}^{j}|C_{j,m}|^2. \tag{48}$$

As in our model $j=n/2$ is a fixed and finite number, the states (46) are only of partial coherence and they correspond to a theory which is entirely based on the ability of particle fields to interfere with each other. It has again to be emphasized that this result depends on our definition how particles can be detected and how correlation functions can be determined experimentally.

It is the formal result of this chapter that this finite dimensional representation of the $sl(2, R)$ algebra is *not a* regular coherent representation as it results only in coherent states of partial coherence.

4. Coherent states and the algebra $su(1, 1)$

We mentioned already at the end of the introductory part that it is also possible to define step operators of the $sl(2, R)$ algebra in relation to a representation which is only bounded below and to calculate then the coherent states using either equation (9) or equation (3). We demonstrate this by using earlier results of Barut and Girardello [5], and investigate the $su(1, 1)$ algebra

$$[L_0, L_\pm]=\pm L_\pm \qquad [L_+, L_-]=-L_0 \tag{49}$$

which has a real isomorphism to the $sl(2, R)$ algebra, namely

$$J_\pm\Rightarrow\pm\sqrt{2}\,L_\pm \qquad J_0\Rightarrow L_0. \tag{50}$$

($J_\pm$ and J_0 are the operators of equation (27).) Barut and Girardello [5] calculated the complete set of eigenstates of equation (3) in an irreducible representation of the $su(1, 1)$ algebra which was bounded only below. The connection to Glauber's definition of coherence is established by calculating the coherent states which follow from the solution of Glauber's factorization rule equation (9) using the same representation.

Thus, we define the eigenstate $|l\rangle$ of the L_0 operator

$$L_0|l\rangle=l|l\rangle \tag{51}$$

and it becomes immediately apparent from equation (49) that the operators $L_+(L_-)$ are shift operators which increase (decrease) the eigenvalue l by one. If we denote the smallest value of l by ϕ, we get

$$\phi+l=\bar{m} \tag{52}$$

with the integer number $\bar{m}$. Thus the spectrum $\bar{m}$ is discrete and bounded below.

We define new states $|\phi, \bar{m}\rangle$ and describe the action of the L-operators by

$$L_{+}|\phi, \bar{m}\rangle = \frac{1}{\sqrt{2}}\sqrt{(\bar{m}+1)(\bar{m}-2\phi)}\,|\phi, \bar{m}+1\rangle$$

$$L_{-}|\phi, \bar{m}\rangle = \frac{1}{\sqrt{2}}\sqrt{\bar{m}(\bar{m}-1-2\phi)}\,|\phi, \bar{m}-1\rangle$$

$$L_0|\phi, \bar{m}\rangle = (\bar{m}-\phi)|\phi, \bar{m}\rangle \tag{53}$$

which corresponds to the discrete (infinite dimensional)$D^{+}(\phi)$ representation of the $su(1, 1)$ algebra [14].

Glauber's factorization rule

$$_q\langle|\overbrace{^{q\ldots q}}^{s}|\rangle\overbrace{^{q\ldots q}_{q}}^{s} = \{_q\langle|^q|\rangle^q_q\}^s \qquad s=1, 2, \ldots, \infty \tag{54}$$

differs from equation (36) only in the fact that s is now only bounded below. We introduce states

$$|\rangle_q = \sum_{\bar{m}=0}^{\infty} C_{\bar{m}}|\phi, \bar{m}\rangle_q \tag{55}$$

and find with $f_1(\phi, \bar{m}) = \sqrt{(\bar{m}-1-2\phi)/2}$:

$$_q\langle|^q|\rangle^q_q = \sum_{\bar{m}=1}^{\infty} |C_{\bar{m}}|^2 \bar{m} f_1^2(\phi; \bar{m}) \tag{56}$$

and

$$_q\langle|\overbrace{^{q\ldots q}}^{s}|\rangle\overbrace{^{q\ldots q}_{q}}^{s} = \sum_{\bar{m}=s}^{\infty} |C_{\bar{m}}|^2 \prod_{\nu=0}^{s-1} (\bar{m}-\nu) f_1^2(\phi; \bar{m}-\nu) \tag{57}$$

as a consequence of equations (53) and (55). As a result, the coherence condition (54) transforms into a system of equations linear in $|C_{\bar{m}}|^2$:

$$\sum_{\bar{m}=s}^{\infty} |C_{\bar{m}}|^2 \prod_{\nu=0}^{s-1} (\bar{m}-\nu) f_1^2(\phi; \bar{m}-\nu) = \alpha^{2s} \qquad s=1, 2, \ldots, \infty \tag{58}$$

which can be solved recursivly in comparing line s to line $s+1$:

$$|C_{\bar{m}+1}|^2(\bar{m}+1) f_1^2(\phi; \bar{m}+1) = \alpha^2 |C_{\bar{m}}|^2 \tag{59}$$

or

$$C_{\bar{m}} = \frac{\alpha^{\bar{m}}}{\sqrt{\bar{m}!}} \frac{C_0}{\prod_{\nu=1}^{\bar{m}} f_1(\phi; \nu)}$$

$$= \frac{(\sqrt{2}\alpha)^{\bar{m}} C_0 \Gamma^{1/2}(-2\phi)}{\sqrt{\bar{m}!}\,\Gamma^{1/2}(\bar{m}-2\phi)}. \tag{60}$$

As a result we find the states of complete coherence $|\alpha\rangle_q$ with:

$$|\alpha\rangle_q = C_0 \sum_{\bar{m}=0}^{\infty} \frac{(\sqrt{2}\alpha)^{\bar{m}} \Gamma^{1/2}(-2\phi)}{\sqrt{\bar{m}!}\,\Gamma^{1/2}(\bar{m}-2\phi)} |\bar{m}\rangle_q \tag{61}$$

with C_0 determined from the renormalization condition

$$1 = |C_0|^2 \Gamma(-2\phi) \sum_{\bar{m}=0}^{\infty} \frac{(2|\alpha|^2)^{\bar{m}}}{\bar{m}!\,\Gamma(\bar{m}-2\phi)}. \tag{62}$$

These states are neither orthogonal nor linearly independent. In passing we would like to note that these coherent states are certainly different from those one would obtain in the limit of equation (12) in connection with equation (9) as those would then be based on a Heisenberg algebra.

A comparison of equations (61) and (62) with the coherent states calculated by Barut and Girardello [5] by explicitely solving the eigenvalue equation

$$L_-|\alpha\rangle = \alpha|\alpha\rangle \tag{63}$$

proves that their solution is identical to ours. Thus, the $D^+(\phi)$ representation of the $su(1,1)$ algebra is a *regular coherent* representation. This is also valid for the representation of the $sl(2,R)$ algebra which is isomorphic to the $D^+(\phi)$ representation.

Consequently, a regular coherent representation of a rank one Lie algebra is found in all cases where coherent states which were obtained from the application of Glauber's factorization rule on one pair of creation and annihilation operators are proportional to eigenstates of one of these operators. Is, moreover, the representation space spanned by states which allow a physical interpretation, we can also speak of physically meaningful coherent states.

Accordingly, the finite dimensional representation of the $sl(2,R)$ algebra used by Ledinegg and Schachinger [11] to describe coherent neutron fields allowed a physical interpretation but did not result in a regular coherent representation (i.e. only coherence of n-th order). On the other hand, the infinite dimensional representation of the $su(1,1)$ algebra (and its isomorphic equivalent of the $sl(2,R)$ algebra) was proved to be regular coherent but the state space was spanned by vectors which could not be interpreted physically. In both cases, the definition of the physical properties of the particle detectors used in the experiment played a critical role in the argument.

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